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Parametric Excitation of Computational Mode of the Leapfrog Scheme Applied to the Van der Pol Equation

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The leapfrog scheme is applied to the Van der Pol equation. When the amplitude of oscillation of the physical mode exceeds a critical value, the computational mode is parametrically excited by the physical mode. The growth of the computational mode interrupts the integration based on the leapfrog scheme. The critical amplitude of the physical mode is determined by the linear stability analysis and the parametric excitation theory. The Runge–Kutta smoother eliminating the computational mode enables the longtime integration based on the leapfrog scheme. © 1993 Academic Press, Inc.

1. INTRODUCTION

The leapfrog scheme used for solving differential equations is cpu-time-saving compared to another scheme such as the Runge–Kutta scheme. The defect of the leapfrog scheme is that it causes the computational mode [1–4]. The computational mode once excited grows with time in a manner of $(-1)^n F^n$, where n denotes the time step and $|F^n|$ increases with n monotonically (see Appendix). The growth of the computational mode interrupts the integration based on the leapfrog scheme.

In our previous paper [5] for the longtime integration of the Korteweg–de Vries equation by the use of the leapfrog scheme, we showed that the physical mode parametrically excited the computational mode. In the subsequent paper [6], we proposed the Runge–Kutta smoother which can successfully eliminate the computational mode. The Runge–Kutta smoother combined with the leapfrog scheme enables us to perform the longtime integration of the Korteweg–de Vries equation.

In this paper, we apply the leapfrog scheme to the Van der Pol equation. We show that the growth of the computational mode interrupts the integration of the Van der Pol equation. The mechanism of parametric excitation of the computational mode by the physical mode is clarified. The critical amplitude of the physical mode is determined. In order to perform the longtime integration based on the leap-frog scheme, elimination of the computational mode is indispensable. A way of eliminating the computational mode is proposed.

2. LEAPFROG SCHEME FOR THE VAN DER POL EQUATION

If we apply the multiple-time-scale perturbation analysis to the Van der Pol equation

$$\frac{d^2x}{dt^2} - 2\varepsilon(1-x^2)\frac{dx}{dt} + x = 0 \qquad (0<\varepsilon\ll 1), \tag{1}$$

we obtain the approximate solution as

$$x(t) = \frac{2\cos(t+\phi)}{[1+(4/X_0^2-1)\,e^{-2\varepsilon t}]^{1/2}}, \tag{2}$$

where X_0 and ϕ are constants [7]. We see from Eq. (2) that

$$x(t) \simeq X_0\cos(t+\phi), \qquad \text{when} \quad 0 \leqslant t \ll 1/\varepsilon$$

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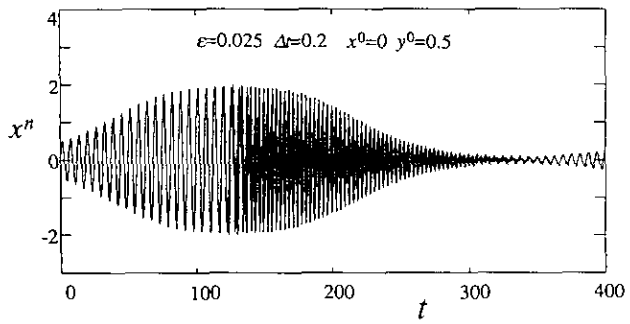

FIG. 1. Solution of leapfrog schemes (6) and (7).

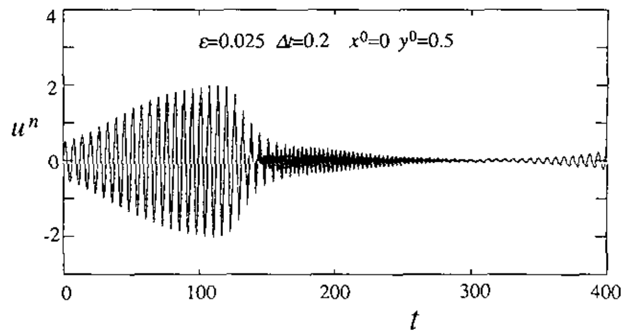

FIG. 3. The physical mode u^n of x^n in Fig. 1.

and

$$x(t) \simeq 2\cos(t+\phi), \qquad \text{when} \quad 1/\varepsilon \ll t. \tag{3}$$

Thus, the amplitude of sinusoidal oscillation varies slowly from X_0 to 2.

The Van der Pol equation (1) is written as simultaneous first-order differential equations as

$$\frac{dx}{dt} = y, \tag{4}$$

$$\frac{dy}{dt} = 2\varepsilon(1-x^2)\,y - x. \tag{5}$$

The leapfrog scheme for Eqs. (4) and (5) gives

$$\frac{x^{n+1}-x^{n-1}}{2\Delta t} = y^n, \tag{6}$$

$$\frac{y^{n+1}-y^{n-1}}{2\Delta t} = 2\varepsilon(1-x^{n2})\,y^n - x^n, \tag{7}$$

where $x^n = x(t = n\,\Delta t)$ and $y^n = y(t = n\,\Delta t)$.

Figure 1 gives x^n obtained from Eqs. (6) and (7) for $\varepsilon = 0.025$ and $\Delta t = 0.2$. The initial values are $x^0 = 0$ and $y^0 = 0.5$. Then x^1 and y^1 are calculated by the Euler scheme with the time increment Δt. Thus, we fail to get the asymptotic solution (3) with the constant amplitude of two when $t \to \infty$. Figure 2 gives x^n in Fig. 1 for $t = 80 \sim 120$. The figure shows the onset of the computational mode oscillating in a manner of $(-1)^n$. We decompose x^n and y^n into physical and computational modes as

$$\begin{aligned} x^n &= u^n + (-1)^n v^n, \\ y^n &= \xi^n + (-1)^n \eta^n, \end{aligned} \tag{8}$$

where u^n and ξ^n are the physical modes and $(-1)^n v^n$ and $(-1)^n \eta^n$ are the computational modes. In order to determine u^n, v^n, ξ^n, and η^n from x^n and y^n, we integrate Eqs. (4) and (5) from $t = (n-1)\,\Delta t$ to $t = n\,\Delta t$, using the Runge–Kutta scheme, and obtain $x(t = n\,\Delta t)$ and $y(t = n\,\Delta t)$ from x^{n-1} and y^{n-1} [5, 6]. Then u^n, v^n, ξ^n, and η^n are given by

$$\begin{aligned} u^n &= \tfrac{1}{2}\,[x^n + x(n\,\Delta t)], \\ (-1)^n v^n &= \tfrac{1}{2}\,[x^n - x(n\,\Delta t)], \\ \xi^n &= \tfrac{1}{2}\,[y^n + y(n\,\Delta t)], \\ (-1)^n \eta^n &= \tfrac{1}{2}\,[y^n - y(n\,\Delta t)]. \end{aligned} \tag{9}$$

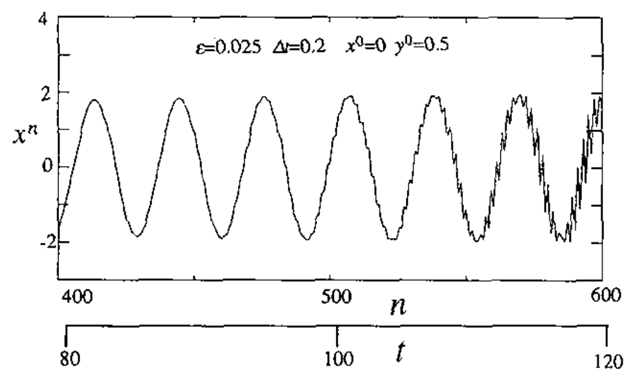

FIG. 2. The onset of the computational mode. Magnified figure of Fig. 1.

Figures 3 and 4 give the physical mode u^n and the computational mode v^n of x^n in Fig. 1. These figures show that the onset of the computational mode v^n interrupts the correct

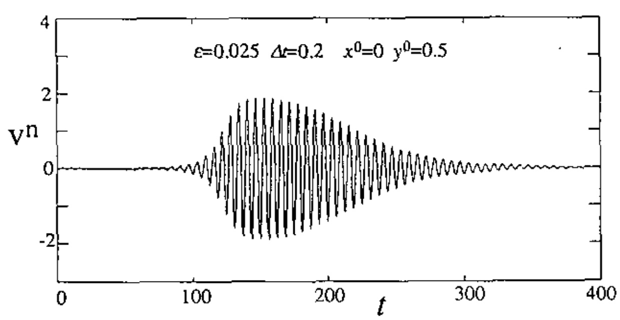

FIG. 4. The computational mode v^n (not $(-1)^n v^n$) of x^n in Fig. 1.

temporal development of the physical mode u^n. Figures 3 and 4 also show that the computational mode v^n starts to grow when the amplitude of the physical mode u^n exceeds a certain threshold, whose value will be examined later.

3. LINEAR ANALYSIS OF LEAPFROG SCHEME

We eliminate y^{n-1}, y^n, and y^{n+1} in Eq. (7) using Eq. (6) to obtain

$$\frac{x^{n+2}-2x^n+x^{n-2}}{4\Delta t^2}=\varepsilon(1-x^{n2})\frac{x^{n+1}-x^{n-1}}{\Delta t}-x^n. \quad (10)$$

We replace $1-x^{n2}$ in the right-hand side of Eq. (10) by $1-X^2$, where X is constant. Then we obtain the linear version of Eq. (10) as

$$\frac{x^{n+2}-2x^n+x^{n-2}}{4\Delta t^2}=\varepsilon(1-X^2)\frac{x^{n+1}-x^{n-1}}{\Delta t}-x^n. \quad (11)$$

Equation (11) has the solution of the form

$$x^n \propto e^{in\tau}, \quad (12)$$

where $i=\sqrt{-1}$. Substitution of Eq. (12) into Eq. (11) gives

$$\sin^2\tau+2i\varepsilon\,\Delta t(1-X^2)\sin\tau-\Delta t^2=0$$

or

$$\sin\tau=-i(1-X^2)\,\varepsilon\,\Delta t\pm\sqrt{1-\varepsilon^2(1-X^2)^2}\,\Delta t. \quad (13)$$

Since $\varepsilon \ll 1$, the value in the square root is positive. If we put

$$\tau=\tau_R+i\tau_I,$$

then we obtain from Eq. (13)

$$\sin\tau_R\cosh\tau_I=\pm\sqrt{1-\varepsilon^2(1-X^2)^2}\,\Delta t, \quad (14)$$

$$\cos\tau_R\sinh\tau_I=-(1-X^2)\,\varepsilon\,\Delta t. \quad (15)$$

If we assume that $|\tau_I| \ll 1$, then $\cosh\tau_I \simeq 1$ so that Eq. (14) gives

$$\tau_R=\pm\sqrt{1-\varepsilon^2(1-X^2)^2}\,\Delta t \quad (16)$$

or

$$\tau_R=\pi\pm\sqrt{1-\varepsilon^2(1-X^2)^2}\,\Delta t. \quad (17)$$

Equation (15) with Eq. (16) gives

$$\tau_I=-\varepsilon(1-X^2)\,\Delta t. \quad (18)$$

Equation (15) with Eq. (17) gives

$$\tau_1=\varepsilon(1-X^2)\,\Delta t. \quad (19)$$

The general solution of Eq. (11) is given by Eqs. (12), (16)–(19) as

$$\begin{aligned} x^n={}&e^{\varepsilon(1-X^2)n\Delta t}[A\cos\sqrt{1-\varepsilon^2(1-X^2)^2}\,n\,\Delta t\\ &+B\sin\sqrt{1-\varepsilon^2(1-X^2)^2}\,n\Delta t]\\ &+(-1)^n e^{-\varepsilon(1-X^2)n\Delta t}[C\cos\sqrt{1-\varepsilon^2(1-X^2)^2}\,n\,\Delta t\\ &+D\sin\sqrt{1-\varepsilon^2(1-X^2)^2}\,n\,\Delta t], \end{aligned} \quad (20)$$

where A, B, C, and D are constants. The first and second terms of the right-hand side of Eq. (20) represent the physical and computational modes, respectively. The computational mode becomes unstable when

$$|X|>1. \quad (21)$$

A more simple second-order finite difference equation corresponding to the Van der Pol equation (1) is

$$\frac{x^{n+1}-2x^n+x^{n-1}}{\Delta t^2}-\varepsilon(1-X^2)\frac{x^{n+1}-x^{n-1}}{\Delta t}+x^n=0, \quad (22)$$

where we have put $1-x^{n2}=1-X^2$. It is interesting to note that the first term of the righ-hand side of Eq. (20) is the general solution of Eq. (22). Then Eq. (22) gives no computational mode.

4. PARAMETRIC EXCITATION OF COMPUTATIONAL MODE

If we substitute x^n in Eq. (8) into Eq. (10), we obtain

$$\theta^n+(-1)^n\delta^n=0, \quad (23)$$

where

$$\begin{aligned} \theta^n \equiv{}& \frac{u^{n+2}-2u^n+u^{n-2}}{4\Delta t^2}-\varepsilon(1-u^{n2}-v^{n2})\\ &\times\frac{u^{n+1}-u^{n-1}}{\Delta t}+u^n-2\varepsilon u^n v^n\frac{v^{n+1}-v^{n-1}}{\Delta t}, \end{aligned} \quad (24)$$

$$\begin{aligned} \delta^n \equiv{}& \frac{v^{n+2}-2v^n+v^{n-2}}{4\Delta t^2}+\varepsilon(1-u^{n2}-v^{n2})\\ &\times\frac{v^{n+1}-v^{n-1}}{\Delta t}+v^n+2\varepsilon u^n v^n\frac{u^{n+1}-u^{n-1}}{\Delta t}. \end{aligned} \quad (25)$$

Since u^n and v^n do not contain the temporal variation expressed by $(-1)^n$, we assume that θ^n and δ^n change smoothly with n. Then θ^n and δ^n can be expanded as

$$\theta^{n+1} = \theta^n + O(\Delta t) \qquad \text{and} \qquad \delta^{n+1} = \delta^n + O(\Delta t), \tag{26}$$

respectively. If we substitute Eq. (26) into $\theta^{n+1} - (-1)^n \delta^{n+1} = 0$, we obtain

$$\theta^n + O(\Delta t) = (-1)^n [\delta^n + O(\Delta t)]. \tag{27}$$

Substitution of Eq. (23) into Eq. (27) gives

$$\theta^n = O(\Delta t) \qquad \text{and} \qquad \delta^n = O(\Delta t). \tag{28}$$

Therefore, θ^n and δ^n in Eq. (23) may be set equal to zero for a sufficiently small Δt. Thus, we obtain

$$\frac{u^{n+2} - 2u^n + u^{n-2}}{4\Delta t^2} - \varepsilon(1 - u^{n2} - v^{n2}) \frac{u^{n+1} - u^{n-1}}{\Delta t} + u^n = 2\varepsilon u^n v^n \frac{v^{n+1} - v^{n-1}}{\Delta t}, \tag{29}$$

$$\frac{v^{n+2} - 2v^n + v^{n-2}}{4\Delta t^2} + \varepsilon(1 - u^{n2} - v^{n2}) \frac{v^{n+1} - v^{n-1}}{\Delta t} + v^n = -2\varepsilon u^n v^n \frac{u^{n+1} - u^{n-1}}{\Delta t}. \tag{30}$$

If we consider the initial stage of the growth of the computational mode $v^n (|v^n| \ll |u^n|)$, we can neglect the right-hand side of Eq. (29) and v^{n2} in the left-hand side of Eqs. (29) and (30). Then we write Eqs. (29) and (30) in forms of differential equations as

$$\frac{d^2u}{dt^2} - 2\varepsilon(1 - u^2) \frac{du}{dt} + u = 0, \tag{31}$$

$$\frac{d^2v}{dt^2} + 2\varepsilon \frac{dv}{dt} + v = 2\varepsilon u \left(u \frac{dv}{dt} - 2v \frac{du}{dt} \right). \tag{32}$$

Equation (31) is the Van der Pol equation which has the solution of the form

$$u = U \cos t = \tfrac{1}{2} U(e^{it} + e^{-it}). \tag{33}$$

U is a slowly varying function of t. We put

$$v = Ve^{it} + V^* e^{-it}, \tag{34}$$

where V is the slowly varying function of t, and V^* is the complex conjugate of V. We substitute Eqs. (33) and (34) into Eq. (32), assume that $dU/dt \simeq 0\,(\varepsilon U)$, $dV/dt \simeq 0\,(\varepsilon V)$, and neglect terms of order ε^2 of Eq. (32). Then we obtain

$$\begin{aligned} &\frac{dV}{dt} e^{it} - \frac{dV^*}{dt} e^{-it} + \varepsilon(Ve^{it} - V^* e^{-it}) \\ &= \frac{1}{4} \varepsilon U^2 (e^{2it} + 2 + e^{-2it})(Ve^{it} - V^* e^{-it}) \\ &\quad - \frac{1}{2} \varepsilon U^2 (e^{2it} - e^{-2it})(Ve^{it} + V^* e^{-it}). \end{aligned} \tag{35}$$

We collect terms only containing e^{it} and e^{-it} to obtain

$$\left. \begin{aligned} \frac{dV}{dt} - \varepsilon \left(\frac{U^2}{2} - 1 \right) V &= -\frac{3}{4} \varepsilon U^2 V^* \\ \frac{dV^*}{dt} - \varepsilon \left(\frac{U^2}{2} - 1 \right) V^* &= -\frac{3}{4} \varepsilon U^2 V \end{aligned} \right\}, \tag{36}$$

respectively. Equation (36) has the solution of the form

$$V, V^* \propto e^{\gamma t},$$

where γ is given by

$$\begin{vmatrix} \gamma - \varepsilon \left(\frac{U^2}{2} - 1 \right) & \frac{3}{4} \varepsilon U^2 \\ \frac{3}{4} U^2 & \gamma - \varepsilon \left(\frac{U^2}{2} - 1 \right) \end{vmatrix} = 0.$$

Then the growth rate γ of the computational mode is given by

$$\gamma = \tfrac{5}{4} \varepsilon (U^2 - \tfrac{4}{5}) \tag{37}$$

or

$$\gamma = -\varepsilon (\tfrac{1}{4} U^2 + 1).$$

From Eq. (37), the computational mode is unstable when

$$|U| > \sqrt{4/5} = 0.894. \tag{38}$$

Inequality (38) is the improved instability condition of Eq. (21). In Figs. 3 and 4, it is not so easy to find the clear onset point of the computational mode instability, because the initial value of the computational mode $|v^n|$ is too small ($\sim 10^{-8}$). If we see the logarithmic plot of $|v^n|$ in Fig. 7 (whose explanation will be given later), it may be found that $|v^n|$ starts to grow around $t \simeq 30$. The physical mode threshold then is found to be about one, from Fig. 3. This may confirm the instability condition (38).

In this section, we solved the differential equations (31) and (32) which hold true only when $|v^n| \ll |u^n|$ in the difference equations (29) and (30). The full differential versions of Eqs. (29) and (30) are

$$\frac{d^2u}{dt^2} - 2\varepsilon(1-u^2-v^2)\frac{du}{dt} + u = 4\varepsilon uv\frac{dv}{dt}, \quad (39)$$

$$\frac{d^2v}{dt^2} + 2\varepsilon(1-u^2-v^2)\frac{dv}{dt} + v = -4\varepsilon uv\frac{du}{dt}. \quad (40)$$

In order to check the validity of setting θ^n and δ^n to be zero in Eq. (23), we solved Eqs. (39) and (40) by the Runge–Kutta scheme with a small initial value of v. Equations (39) and (40) were found to lead us essentially to the same results as shown in Figs. 3 and 4. It is interesting to note that Eqs. (39) and (40) coincide, respectively, with (A8a) and (A8b) in the Appendix, which are reducible from the so-called augmented system introduced by J. M. Sanz-Serna *et al.* [2, 3].

5. RUNGE–KUTTA SMOOTHER

We integrate Eqs. (4) and (5) from $t=(n-1)\Delta t$ to $t=n\Delta t$ using the Runge–Kutta scheme to obtain $x(t=n\Delta t)$ and $y(t=n\Delta t)$ from x^{n-1} and y^{n-1}. We also integrate inversely Eqs. (4) and (5) from $t=n\Delta t$ to $t=(n-1)\Delta t$ by the use of the Runge–Kutta scheme to obtain $x((n-1)\Delta t)$ and $y((n-1)\Delta t)$ from x^n and y^n.

The physical modes u^{n-1}, u^n, ξ^{n-1}, and ξ^n are given by

$$u^{n-1} = \tfrac{1}{2}[x^{n-1} + x((n-1)\Delta t)], \quad u^n = \tfrac{1}{2}[x^n + x(n\Delta t)],$$
$$\xi^{n-1} = \tfrac{1}{2}[y^{n-1} + y((n-1)\Delta t)], \quad \xi^n = \tfrac{1}{2}[y^n + y(n\Delta t)]. \quad (41)$$

We start the leapfrog schemes (6) and (7) using u^{n-1}, u^n, ξ^{n-1}, and ξ^n in place of x^{n-1}, x^n, y^{n-1}, and y^n, respectively. Then we can eliminate the computational modes.

We apply the Runge–Kutta smoother noted above to the case of Fig. 1 at every 40 time steps. Figures 5 and 6 give the physical mode u^n and the computational mode v^n of x^n. Thus, the Runge–Kutta smoother enables the correct longtime integration. We also have applied the second-order smoother in place of the Runge–Kutta smoother and found it is not as effective as the Runge–Kutta smoother. To compare the second-order smoother with the Runge-Kutta smoother, see Ref. [6].

Figure 7 gives the computational mode $|v^n|$ of x^n obtained from Eqs. (6) and (7). The Runge–Kutta smoother is applied at $t=100$. For $t \simeq 100 \sim 380$, the amplitude of the physical mode u^n, which is not shown in the figure, is nearly equal to two. If we put $U=2$ and $\varepsilon=0.025$ in Eq. (37) we obtain $\gamma=0.1$. We gave the line $|v^n| \propto e^{0.1t}$ in Fig. 7. Thus, the theory of the parametric excitation in Section 4 successfully explains the growth of the computational mode.

The discussion in this paper is based on the assumption that the leapfrog solutions are expressed by a sum of physical and computational modes as in the linear case. We find that this assumption can be verified naturally by investigating the process leading to the augmented system associated with the leapfrog scheme as shown in the Appendix. In the augmented system, the information of the change of computational modes expressed by $(-1)^n$ has been lost due to the limit operation of $\Delta t \to 0$. Our quite practical scheme to suppress the leapfrog instability or the Runge–Kutta smoother is simply based on the fact that the computational modes exhibit the change expressed by $(-1)^n$, which might not be reduced from any investigation of the augmented system alone.

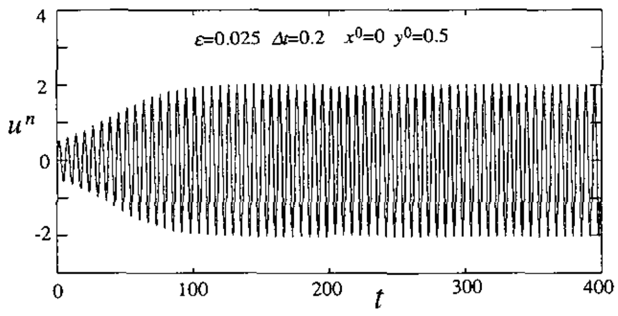

FIG. 5. The physical mode u^n of x^n in Fig. 1. The Runge–Kutta smoother is applied at $t=40\times$integers.

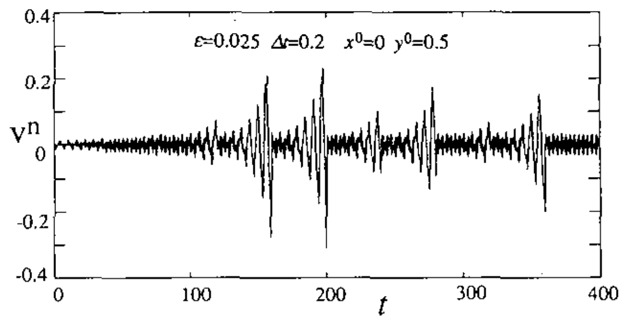

FIG. 6. The computational mode v^n corresponding to Fig. 5.

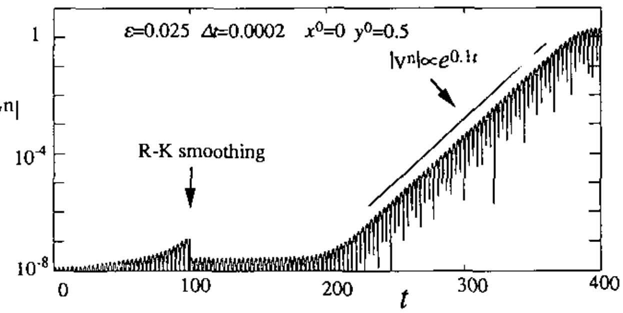

FIG. 7. The growth of the computational mode $|v^n|$.

APPENDIX

J. M. Sanz-Serna *et al.* [2, 3] studied the temporal behavior of the numerical solution of leapfrog schemes in terms of the so-called augmented system. The augmented system was shown to describe the qualitative behavior of the leapfrog solution, in particular, the nonlinear instability phenomenon.

In this Appendix, we show that the leapfrog solutions can be expressed by a sum of the physical and computational modes and the latter modes change their signs at each time step as in the linear case.

We consider a system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(x), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{A1}$$

where the function $\mathbf{F}$ is smooth and $\mathbf{x}_0$ is the initial value. The leapfrog discretization of Eq. (A1) with the time increment Δt gives

$$\frac{\mathbf{x}^{n+1} - \mathbf{x}^{n-1}}{2\Delta t} = \mathbf{F}(\mathbf{x}^n), \qquad \mathbf{x}^0 = \mathbf{x}_0,\ \mathbf{x}^1 = \mathbf{x}_1, \tag{A2}$$

where the starting value $\mathbf{x}_1$ is assumed to be given.

If we divide the sequence of $\mathbf{x}^n$ into even and odd groups, then they can be expressed by

$$\mathbf{x}^{2m} = \mathbf{p}^{2m} \qquad \text{and} \qquad \mathbf{x}^{2m+1} = \mathbf{q}^{2m+1}, \tag{A3}$$

respectively. Now Eq. (A2) can equivalently be expressed by

$$\frac{\mathbf{p}^{2m} - \mathbf{p}^{2m-2}}{2\Delta t} = \mathbf{F}(\mathbf{q}^{2m-1}), \qquad \mathbf{p}^0 = \mathbf{x}_0, \tag{A4a}$$

$$\frac{\mathbf{q}^{2m+1} - \mathbf{q}^{2m-1}}{2\Delta t} = \mathbf{F}(\mathbf{p}^{2m}), \qquad \mathbf{q}^1 = \mathbf{x}_1. \tag{A4b}$$

In the limit of $\Delta t \to 0$ with $n\,\Delta t$ fixed, Eqs. (A4) lead to the augmented system for Eq. (A2) [2, 3]:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{q}), \quad \mathbf{p}(0) = \mathbf{x}_0, \tag{A5a}$$

$$\frac{d\mathbf{q}}{dt} = \mathbf{F}(\mathbf{p}), \quad \mathbf{q}(0) = \mathbf{x}_1. \tag{A5b}$$

If $\mathbf{x}_1 = \mathbf{x}_0$ in Eq. (A5), then we would have a solution $\mathbf{p}(t) = \mathbf{q}(t) = \mathbf{x}(t)$. In this case, the augmented system (A5) coincides with the original system (A1). In the usual case, however, $\mathbf{x}_1 \neq \mathbf{x}_0$ for the finite Δt in (A2), so that the augmented system gives a solution $\mathbf{p}(t) \neq \mathbf{q}(t)$, either of which is also different from $\mathbf{x}(t)$.

Now let us note that Eq. (A3) can be expressed formally as

$$\mathbf{x}^n = \tfrac{1}{2}(\mathbf{p}^n + \mathbf{q}^n) + (-1)^n \tfrac{1}{2}(\mathbf{p}^n - \mathbf{q}^n). \tag{A6}$$

According to J. M. Sanz-Serna [2], $(\mathbf{p}^n, \mathbf{q}^{n+1})$ follow approximately the local solution $(\mathbf{p}(t), \mathbf{q}(t))$ of the augmented system (A5). Therefore, we can approximate $\mathbf{p}^n$ and $\mathbf{q}^n$ in Eq. (A6) by

$$\mathbf{p}^n \simeq \mathbf{p}(n\,\Delta t) \qquad \text{and} \qquad \mathbf{q}^n \simeq \mathbf{q}((n-1)\,\Delta t),$$

respectively. By setting

$$\mathbf{u}^n = \tfrac{1}{2}(\mathbf{p}^n + \mathbf{q}^n) \qquad \text{and} \qquad \mathbf{v}^n = \tfrac{1}{2}(\mathbf{p}^n - \mathbf{q}^n),$$

we can rewrite Eq. (A6) as

$$\mathbf{x}^n = \mathbf{u}^n + (-1)^n \mathbf{v}^n. \tag{A7}$$

The first and second terms in the right-hand side of Eq. (A7) are, respectively, the physical and computational modes, which are in the same form as those in the linear case. The evolution equations for both modes are derived from Eq. (A5) and they are

$$\frac{d\mathbf{u}}{dt} = \frac{1}{2}\,[\mathbf{F}(\mathbf{u}-\mathbf{v}) + \mathbf{F}(\mathbf{u}+\mathbf{v})] \tag{A8a}$$

and

$$\frac{d\mathbf{v}}{dt} = \frac{1}{2}\,[\mathbf{F}(\mathbf{u}-\mathbf{v}) - \mathbf{F}(\mathbf{u}+\mathbf{v})], \tag{A8b}$$

respectively.

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